14 *Years*

Previous Years Solved Papers

Civil Services Main Examination

(2009-2022)

Mathematics

Paper-II

Topicwise Presentation

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Civil Services Main Examination Previous Solved Papers : Mathematics Paper-II

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Preface

B. Singh (Ex. IES)

Civil Service is considered as the most prestigious job in India and it has become a preferred destination by all engineers. In order to reach this estimable position every aspirant has to take arduous journey of Civil Services Examination (CSE). Focused approach and strong determination are the pre-requisites for this journey. Besides this, a good book also comes in the list of essential commodity of this odyssey.

I feel extremely glad to launch the fourth edition of such a book which will not only make CSE plain sailing, but also with 100% clarity in concepts.

MADE EASY team has prepared this book with utmost care and thorough study of all previous years papers of CSE. The book aims to provide complete solution to all previous years questions with accuracy.

I would like to acknowledge efforts of entire MADE EASY team who worked day and night to solve previous years papers in a limited time frame and I hope this book will prove to be an essential tool to succeed in competitive exams and my desire to serve student fraternity by providing best study material and quality guidance will get accomplished.

With Best Wishes

B. Singh (Ex. IES)

CMD, MADE EASY Group

Previous Years Solved Papers of

Civil Services Main Examination

Mathematics : Paper-II

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1 Modern Algebra

1. Groups

1.1 If $\mathbb{R}$ is the set of real number and $\mathbb{R}_+$ is the set of positive real numbers, show that $\mathbb{R}$ under addition ($\mathbb{R}$, +) and $\mathbb{R}_+$ under multiplication ($\mathbb{R}$, +) are isomorphic. Similarly, if $\mathbb{Q}$ is the set of rational numbers and $\mathbb{Q}_+$ the set of positive rational numbers, are ($\mathbb{Q}$, +) and ($\mathbb{Q}_+$, +) isomorphic? Justify your answer.

(2009 : 4+8=12 Marks)

Solution:

Let $\mathbb{R}$ be the set of real number and $\mathbb{R}_+$ be the set of positive real number.

We have to show

$$(\mathbb{R},+) \cong (\mathbb{R}_+,\cdot)$$

Define $f: \mathbb{R} \to \mathbb{R}_+$ as

$$f(x) = a^x;\text{ where } a > 0.$$

We will show f is one-one.

Consider,

$$\begin{aligned}\ker f &= \{x \in R \mid f(x) = 1\}\\ &= \{x \in \mathbb{R} \mid a^x = 1\}\\ &= \{x \in \mathbb{R} \mid x = \log_a^1\}\\ &= \{x \in \mathbb{R} \mid x = 0\}\\ &= \{0\}\end{aligned}$$

$\therefore$ f is 1 – 1.

We will show f is homomorphism.

Let $x, y \in R$

Consider

$$\begin{aligned}f(x+y) &= a^{x+y}\\ &= a^x.a^y = f(x).f(y)\end{aligned}$$

$\therefore$ f is homomorphism. We will show f is onto, i.e., we have to find for any positive real number 'y' some real number x such that

$$f(x) = y$$

i.e., $a^x = y$

As $a^x = y$

On taking log both sides

$\Rightarrow$ $x = \log_a y$

$\therefore$ $f(x) = y$

Hence, f is onto.

$\therefore$ $(\mathbb{R},+) \cong (\mathbb{R}_+,\cdot)$

Let Q be the set of rational numbers and Q_+ be the set of positive rational number.

If f is homomorphism from Q to Q_+, then

$$f(x, y) = f(x)f(y) \;\forall\; x, y \in Q$$

And if image of 1 is known then the image of every element will be known.

$\therefore$ $f(x) = a^x$ where $a = f(1)$

If $a = 1$, $f(x) = 1$

$\therefore$ f is trivial homomorphism.

If $a \neq 1$, $f(x) = 1$

then $f(x) = a^x \in Q_+ \forall\; x \in Q$

which is a contradiction.

Hence, only trivial homomorphism is possible.

$\therefore$ $(Q, +) \not\cong (Q_+, .)$

1.2 Determine the number of homomorphisms from the additive group Z_{15} to the additive group Z_{10}. (Z_n is the cyclic group of order n).

(2009 : 12 Marks)

Solution:

Let $\phi : \mathbb{Z}_{15} \to \mathbb{Z}_{10}$ be a homomorphism.

As $\mathbb{Z}_{15}$ is a cyclic group of order 15.

$$\mathbb{Z}_{15} = \langle 1 \rangle$$

Under homomorphism, if element 1 will be mapped then remaining elements will get mapped themselves ($\because$ G is cyclic)

Suppose, $\phi(1) = x$

As we know, if f is homomorphism from G to G' then $O(f(a)/0(a)$ where $a \in G$.

As $\phi(1) = x$.

$\therefore$ $O(x)/O(a) = 15$

And order of element divides order of group

$\therefore$ $O(x)/10$ As $O(x)/15$

$O(x)/15 \Rightarrow O(x)|g.c.d(15, 10)O(x)|5$

$\therefore$ $O(x) = 1$ or $O(x) = 5$

If $O(x) = 1$. Then it is trivial homomorphism. And if $O(x) = 5$.

Note : In $\mathbb{Z}_n$, number of elements of order $k = \phi(k)$; provided $k|n$.

$\therefore$ In $\mathbb{Z}_{10}$, number of elements of order $5 = \phi(5) = 4$.

$\therefore$ We have 4 possibilities for x.

Total number of homomorphism = 4 + 1 = 5.

1.3 Show that the alternating group on four letters A_4 has no subgroup of order 6.

(2009 : 15 Marks)

Solution:

Consider the alternating group A_4.

$$\sigma(A_4) = \frac{\dot{\sigma}(S_4)}{2} = \frac{\lfloor 4}{2} = 12$$

We show although $6 \mid 12$, A_4 has no subgroup of order 6. Suppose H is a subgroup of A_4 and $\sigma(H) = 6$.

By previous problem the number of distinct 3-cycles in S_4 is

$$\frac{1}{3} \cdot \frac{4!}{(4-3)!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 1} = 8$$

Again, as each 3-cycle will be even permutation all these 3-cycles are in A_4. Obviously then, at least one 3-cycle, say s does not belong to $H(\sigma(H) = 6)$.

Now, $\sigma \in H \Rightarrow \sigma^2 \in H$, because if $\sigma^2 \in H$.

$$
\begin{aligned}
\text{Then,} \quad \sigma^4 &\in H \\
\Rightarrow \quad \sigma &\in H \\
\text{As} \quad \sigma^3 &= I \text{as } \sigma(\sigma) = 3 \\
\text{Let} \quad K &= \langle\sigma\rangle = \{I, \sigma, \sigma 2\} \text{ then, } \sigma K = 3(= \sigma(\sigma)) \\
\text{and} \quad H \cap K &= \{1\} \qquad (\sigma, \sigma^2 \not\subseteq H) \\
\Rightarrow \quad \sigma(HK) &= \frac{\sigma(H)\cdot\sigma(K)}{\sigma(H \cap K)} = \frac{6.3}{1} = 18,
\end{aligned}
$$

not possible as $HK \subseteq A_4$ and $\sigma(A_4) = 12$.

1.4 Let $G = R - \{-1\}$ be the set of all real numbers omitting -1. Define the binary relation $*$ on G by $a*b = a + b + ab$. Show $(G, *)$ is a group and it is abelian.

(2010 : 12 Marks)

Solution:

Given : Binary relation $*$ as

$$a*b = a + b + ab, \text{ where } a, b \in G.$$

Closure : Let $a, b \in G$

$$\therefore \quad a*b = a + b + ab$$

if $a + b + ab = -1$, then

$$
\begin{aligned}
a + b + ab + 1 &= 0 \\
\Rightarrow \quad (1 + a) + b(1 + a) &= 0 \\
\Rightarrow \quad (1 + a) + (1 + b) &= 0 \\
\Rightarrow \text{either} \quad 1 + a &= 0 \Rightarrow a = -1 \\
\text{or} \quad 1 + b &= 0 \Rightarrow b = -1
\end{aligned}
$$

$\because$ both $a, b \in G$

$$
\begin{aligned}
\therefore \quad a &\neq -1 \text{ and } b \neq -1 \\
\therefore \quad a + b + ab &\neq -1 \text{ for any } a, b \in G \\
\therefore \quad a + b + ab &\neq -1 \text{ for any } a, b \in G \\
\therefore \quad a*b &\in G
\end{aligned}
$$

So, closure is satisfied.

Associative : Let $a, b, c \in G$

$$
\begin{aligned}
\therefore \quad (a*b)*c &= (a + b + ab)*c \\
&= a + b + ab + c + (a + b + ab)c \\
&= a + b + ab + c + ac + bc + abc \\
&= a + b + c + ab + ac + bc + abc \\
\text{Also,} \quad a*(b*c) &= a*(b + c + bc) \\
&= a + b + c + bc + a(b + c + bc) \\
&= a + b + c + ab + bc + ac + abc \\
\text{as} \quad (a*b)*c &= a*(b*c)
\end{aligned}
$$

$\therefore$ Associative property is satisfied.

Identity :

$$\text{Let} \quad a*b = a = a + b + ab$$

$\Rightarrow \quad a + b + ab = a$

$\Rightarrow \quad b(1 + a) = 0$

as $\quad a \neq -1 \Rightarrow b = 0$

$\therefore$ $b = 0$ is an identity and as $O \in G$.

$\therefore$ identity exists.

Inverse :

Let $\quad a*b = 0 = a + b + ab$

$$\Rightarrow \quad a(1 + b) = -b \quad \Rightarrow \quad a = \frac{-b}{1+b} \qquad (b \neq -1)$$

Also, $\quad \dfrac{-b}{1+b} \in G$

$\therefore$ Inverse exists.

As closure, associative property, identity, inverse conditions are satisfied. $\therefore$ $(G, *)$ is a group.

Now, $\quad a*b = a + b + ab$

$\quad b*a = b + a + ba = a + b + ab$

as $\quad a*b = b*a$

$\therefore$ $(G, *)$ is abelian.

1.5 Show that a cyclic group of order 6 is isomorphic to the product of a cyclic group of order 2 and a cyclic group of order 3. Can you generalize this? Justify.

(2010 : 12 Marks)

Solution:

Let z_6 is cyclic group of order 6

z_2 is cyclic group of order 2

z_3 is cyclic group of order 3

Now, we know that

$$Z_m \times Z_n \cong Z_{mn}$$

when m and n are co-prime.

Here, $\quad m = 2, n = 3$

$\therefore \quad z_2 \times z_3 \cong z_6$

1.6 Let $(\mathbb{R}^*, \cdot)$ be the multiplicative group of non-zero real and $((GL(n, \mathbb{R}), X)$ be the multiplicative group of $n \times n$ non-singular real matrices. Show that the quotient group $GL(n, \mathbb{R}) / SL(n, \mathbb{R})$ and $(\mathbb{R}^*, \cdot)$ are isomorphic where

$$SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) / \det A = 1\}$$

What is the center of $GL(n, \mathbb{R})$?

(2010 : 15 Marks)

Solution:

Given, $(GL(n, \mathbb{R}), X)$ is multiplicative group of real matrices.

Let $f : GL(n, \mathbb{R}) \to (\mathbb{R}^*, \cdot)$ be a no morphism where for any $A \in G$.

$$\phi(A) = |A|$$

$$\therefore \quad \phi(A \times B) = |AB| = |A|\,|B|$$

Now, let Ker(ϕ) contains matrices such it $B \in$ Ker (ϕ)

then
$$\phi(AB) = A$$
$$\therefore \quad |B| = 1$$

$\therefore$ Ker(ϕ) contains matrices such that $|B| = 1$ if $B \in$ Ker (ϕ).

$\therefore$ By first fundamental theorem of homomorphism,

$$\frac{GL}{Ker(\phi)} \cong (\mathbb{R}^*, \cdot)$$

Given,
$$SL(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) / |A| = 1\}$$

i.e., $SL(n,\mathbb{R})$ is Ker(ϕ).

$$\therefore \quad \frac{GL(n,\mathbb{R})}{SL(n,\mathbb{R})} \cong (\mathbb{R}^*, \cdot)$$

Now, let z be the center of $GL(n,\mathbb{R})$.

for $x \in z$ where $X = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$

$\forall A \in GL(n,\mathbb{R})$,
$$AX = XA$$

Let
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ where } A \in GL(n,\mathbb{R})$$

Now,
$$AX = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix}$$

$$XA = \begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ap+qc & pb+qd \\ ar+cs & br+ds \end{bmatrix}$$

$\therefore$ for $XA = AX$

$$\begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix} = \begin{bmatrix} ap+qc & pb+qd \\ ar+cs & br+ds \end{bmatrix}$$

Holds true when $p = q = r = s$

or
$$p = s, \; q = r = 0$$

$$\therefore \quad z = \left\{ \begin{bmatrix} p & p \\ p & p \end{bmatrix}, \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} \middle| p \in R \right\}$$

1.7 Show that the set

$$G = \{f_1, f_2, f_3, f_4, f_5, f_6\}$$

of six transformations on the set of Complex numbers defined by

$$f_1(z) = z, \; f_2(z) = 1 - z, \; f_3(z) = \frac{z}{(z-1)},$$

$$f_4(z) = \frac{1}{z}, \; f_5(z) = \frac{1}{(1-z)} \text{ and } f_6(z) = \frac{(z-1)}{z}$$

is a non-abelian group of order 6 w.r.t. composition of mappings.

(2011 : 12 Marks)

Solution:

Let us construct the following table of the elements of G w.r.t. composition of mappings :

	f_1	f_2	f_3	f_4	f_5	f_6
f_1	z	$1-z$	$\frac{z}{z-1}$	$\frac{1}{z}$	$\frac{1}{1-z}$	$\frac{z-1}{z}$
f_2	$1-z$	z	$\frac{1}{1-z}$	$\frac{z-1}{z}$	$\frac{z}{z-1}$	$\frac{1}{z}$
f_3	$\frac{z}{z-1}$	$\frac{z-1}{z}$	z	$\frac{1}{1-z}$	$\frac{1}{z}$	$1-z$
f_4	$\frac{1}{z}$	$\frac{1}{1-z}$	$\frac{z-1}{z}$	z	$1-z$	$\frac{z}{z-1}$
f_5	$\frac{1}{1-z}$	$\frac{1}{z}$	$1-z$	$\frac{z}{z-1}$	$\frac{z-1}{z}$	z
f_6	$\frac{z-1}{z}$	$\frac{z}{z-1}$	$\frac{1}{z}$	$1-z$	z	$\frac{1}{1-z}$

From the table it is clear that

$$(f_i \circ f_i)(z) = (f_i \circ f_i)(z) \ \forall\ z \in C, 1 \le i \le 6$$

$\therefore$ f_i is the identity of G.

Also, $$f_1^{-1} = f_1;\ f_2^{-1} = f_2;\ f_3^{-1} = f_3,\ f_4^{-1} = f_4^1 = f_5^{-1} = f_6;\ f_6^{-1} = f_5$$

$\Rightarrow$ G is a group.

But $$(f_2 \circ f_3)(z) = \frac{1}{1-z}$$

and $$(f_3 \circ f_2)(z) = \frac{z-1}{z}$$

$\therefore$ G is a non-abelian group.

1.8 (i) **Prove that a group of Prime order is abelian.**

(2011 : 6 Marks)

(ii) **How many generators are there of the cyclic group (G, .) of order 8?**

(2011 : 6 Marks)

Solution:

(i) Let $O(G) = P$, where P is a prime number.

Let $a \in h$ be any element.

Since order of an element divides the order of the group.

$\therefore$ $$O(a) \mid O(G) = P$$

$\Rightarrow$ $$O(a) = 1 \text{ or } O(a) = P$$

If $a \neq e$, then $$O(a) = P$$

Let $$H = (a)$$

$\Rightarrow$ $$O(H) = O(a) = P \text{ and } H \subseteq G$$

$\therefore$ $$H = G = (a)$$

$\Rightarrow$ G is cyclic.

$\Rightarrow$ G is abelian **(as every cyclic group is abelian)**.

(ii) Let G be a cyclic group of order 8.

$\therefore$ $\exists$ can element $a \in G$ such that $O(a) = 8$

We know that, $$O(a^n) = \frac{O(a)}{(O(a), n)}$$

$\therefore$ $$O(a^n) = O(a) \Rightarrow (O(a), n) = 1$$

Number of elements of G whose order is co-prime to

$$\begin{aligned} O(a) &= \phi(O(a)) \\ &= \phi(8) = \phi(2^3) \\ &= 4 \qquad (\because \phi(P^n) = P^n - P^{n-1}) \end{aligned}$$

$\therefore$ Number of generators of cyclic group of order 8 = 4.

1.9 Give an example of a group G in which every proper subgroup is cyclic but the group itself is not cyclic.

(2011 : 15 Marks)

Solution:

Let $G = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$

Define product on G by usual multiplication together with

$$\begin{aligned} i^2 &= j^2 = k^2 = -1 \\ ij &= -ji = k \\ jk &= -kj = i \\ ki &= -ik = j \end{aligned}$$

Then G forms a group.

But G is not abelian as $ij \neq ji$.

$\Rightarrow$ G can not be cyclic as every cyclic group is abelian.

Let

$$\begin{aligned} H_1 &= [1] = \langle 1 \rangle \\ H_2 &= [\pm 1] = \langle -1 \rangle \\ H_3 &= [\pm 1, \pm i] = \langle i \rangle = \langle -i \rangle \\ H_4 &= [\pm 1, \pm j] = \langle j \rangle = \langle -j \rangle \\ H_5 &= [\pm 1, \pm k] = \langle k \rangle = \langle -k \rangle \end{aligned}$$

Thus, H_1 to H_5 are proper subgroups of G and all there are cyclic. Hence the result.

1.10 Let a and b be elements of a group, with $a^2 = e$, $b^6 = e$ and $ab = b^4a$. Find the order of ab, and express its inverse in each of the forms $a^m b^n$ and $b^m a^n$.

(2011 : 20 Marks)

Solution:

Given : $a^2 = e,\ b^6 = e$ and $ab = b^4a$

$\therefore$ $ab = b^4a$

$\Rightarrow$ $aba^{-1} = b^4aa^{-1} = b^4e = b^4$

Again,

$$\begin{aligned} b^8 &= b^4 \cdot b^4 \\ &= (aba^{-1})(aba^{-1}) = ab^2a^{-1} \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} b^{16} &= b^8 \cdot b^8 = (ab^2a^{-1})(ab^2a^{-1}) \\ &= ab^4a^{-1} \\ &= a(aba^{-1})a^{-1} \qquad (\because b^4 = aba^{-1}) \\ &= a^2ba^{-2} \\ &= a^2b(a^2)^{-1} \\ &= ebe = b \end{aligned}$$

$\Rightarrow$ $b^{16} = b \Rightarrow b^{15} = e$

$\Rightarrow$ $O(b) = 1, 3, 5 \text{ or } 15$ ...(i)

Also, given $b^6 = e$

$\Rightarrow$ $O(b) = 1, 2, 3 \text{ or } 6$ ...(ii)

$\therefore$ From (i) and (ii)

$$O(b) = 3$$

And $ab = b^4a = b^3ba = eba = ba$

Also, we know that, for $a, b \in G$ such that ab = ba and $(O(a), O(b)) = 1$

then $O(ab) = O(a) \cdot O(b)$

$\therefore$ $O(ab) = 2 \times 3 = 6$

$\Rightarrow$ $(ab)^6 = e$

$\Rightarrow$ $(ab)^5 = (ab)^{-1}$

$\Rightarrow$ $a^5b^5 = (ab)^{-1}$

Similarly, as $ab = ba$, $(ab)^{-1} = b^5a^5$

1.11 How many elements of order 2 are there in the group of order 16 generated by a and b such that the order of a is 8, the order of b is 2 and $bab^{-1} = a^{-1}$.

(2012 : 12 Marks)

Solution:

Let G be the given group of order 16.

Then, $O(G) = 16$

$O(a) = 8$

$O(b) = 2$

and $bab^{-1} = a^{-1}$ ...(i)

From (i), we have

$b(bab^{-1}) = b(a^{-1})$

$\Rightarrow$ $b^2ab^{-1} = ba^{-1}$

$\Rightarrow$ $ab^{-1} = ba^{-1}$ $(\because O(b) = 2 \Rightarrow b^2 = e)$

$\Rightarrow$ $ab = ba^{-1}$ $(\because b^2 = e \Rightarrow b = b^{-1})$

and h is generated by a and b, where $O(a) = 8$, $O(b) = 2$, $bab^{-1} = a^{-1}$.

$\therefore$ $G = D_8$, dihedral group of order 8.

Again, let $H = <a>$

Since, $O(a) = 8$

$\therefore$ H is a cyclic group of order 8. Also, in a finite cyclic group, for each divisor $\frac{k}{n} = O(G)$, there exist exactly $\phi(k)$ elements of order k.

[$\phi(n)$ = positive integers upto a given integer n that are relatively prime to n].

As $2 \mid 8$, $\therefore$ $\phi(2)$ elements of order 2, i.e., 1 element of order 2 $(\because \phi(2) = 1)$

Also, D_8 consists of 8 rotations and 8 reflections.

Since, order of each reflection is 2.

$\therefore$ Number of elements of order 2 = 8 + 1 = 9.

1.12 How many conjugacy classes does the permutation group S_5 of permutations 5 numbers have? Write down one element in each class (preferably in terms of cycles).

(2012 : 15 Marks)

Solution:

We know that two permutations are said to be conjugate if they have same cyclic decomposition.

There are 7 cyclic decompositions which are :

1. 1 + 1 + 1 + 1 + 1 [1 (5 times)]
2. 1 + 1 + 1 + 2 [2 (1 time), 1 (3 times)]
3. 1 + 1 + 3 [1 (2 times), 3 (1 time)]
4. 1 + 4 [1 (1 time), 4 (1 time)]
5. 5 [5 (1 time)]
6. 1 + 2 + 2 [1 (1 time), 2 (2 times)]
7. 2 + 3 [2 (1 time), 3 (1 time)]

$$
\begin{aligned}
\therefore \qquad \text{Number of conjugacy classes} &= 7.\\
\text{An element of 1st class} &= (1)\\
\text{An element of 2nd class} &= (1\ 2)\\
\text{An element of 3rd class} &= (1\ 2\ 3)\\
\text{An element of 4th class} &= (1\ 2\ 3\ 4)\\
\text{An element of 5th class} &= (1\ 2\ 3\ 4\ 5)\\
\text{An element of 6th class} &= (1\ 2)(3\ 4)\\
\text{An element of 7th class} &= (1\ 2\ 3)(4\ 5)
\end{aligned}
$$

1.13 Give an example of an infinite group in which every element has finite order.

(2013 : 10 Marks)

Solution:

Consider $\left\langle \frac{Q}{Z}, t \right\rangle$, i.e., the quotient group of rationals with respect to integers under addition.

Any element of $\frac{Q}{Z}$ is of form $\frac{p}{q} + z$ where $\frac{p}{q} \in Q$ is in lowest form.

$$
\begin{aligned}
q\left(\frac{p}{q} + z\right) &= \underbrace{\frac{p}{q} + \ldots\ldots + \frac{p}{q}}_{q \text{ times}} + z = p + z\\
&= z
\end{aligned}
$$

$$\therefore \qquad O\left(\frac{p}{q} + z\right) \le q$$

So, order of all elements are finite. But there are infinitely elements in $\frac{Q}{Z}$ as there are infinite rationals in [0, 1) and none of them are equal in $\frac{Q}{Z}$.

$$= f(a + ib) + f(c + id)$$

And

$$
\begin{aligned}
f[(a + ib)(c + id)] &= f[ac - bd + i(ad + bc)]\\
&= \begin{bmatrix} ac - bd & -(ad + bc) \\ ad + bc & ac - bd \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c & -d \\ d & c \end{bmatrix}\\
&= f(a + ib) \cdot f(c + id)
\end{aligned}
$$

f is one-one

$$a + ib = c + id \Rightarrow a = c \text{ and } b = d \Rightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} c & -d \\ d & a \end{bmatrix}$$

$$\Rightarrow \qquad f(a + ib) = f(c + id)$$

f is onto

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \in S \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = f(a + ib)$$

$\therefore$ f is a one-one onto linear map and so an isomorphism.

1.14 What are the orders of the following permutations in S_{10}.

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 8 & 7 & 3 & 10 & 5 & 4 & 2 & 6 & 9 \end{pmatrix}$ **and (1 2 3 4 5) (6 7)**

(2013 : 10 Marks)

Solution:

Approach : This and the next part uses the fact that order of any permutation written as disjoint cycles in LCM of the order of the cycle. And order of any cycle is its length.

Writing the permutation as product of disjoint cycles.

(2 8) (3 7 4) (5 10 9 6)

$$\text{Order} = \text{L.C.M.}(2, 3, 4) = 12$$

(1 2 3 4 5)(6 7)

$$\text{Order} = \text{L.C.M.}(5, 2) = 10$$

1.15 What is the maximal possible order of an element in S_{10}? Why? Give an example of that element? How many elements will be there in S_{10} of that order?

(2013 : 13 Marks)

Solution:

Any permutation in S_{10} can be written as product of disjoint cycles. Let $n_1, n_2, \ldots, n_r$ be size of these cycles. Then they are also the order of respective cycles.

$$n_1 + n_2 + \ldots + n_r = 10$$

and $$\text{Order} = \text{L.C.M.}(n_1, n_2, \ldots, n_r)$$

The order will be maximised if n_i's are relative primes.

The primes less than 10 are 2, 3, 5, 7.

Taking $n_1 = 3$, $n_2 = 7$, Order = 21; $n_1 = 2$, $n_2 = 3$, $n_3 = 3$, Order = 30.

∴ The highest order is 30 for cycles of length 2, 3 and 5.

An example : (1, 2)(3 4 5)(6 7 8 9 10)

Number of Such Elements :

${}^{10}C_2$ ways of picking 2 elements for 1st cycle and they can be written $(2-1)! = 1$ way.

${}^{8}C_3$ ways of picking 3 elements from rest 8 elements and can be written in $(3-1)! = 2$ ways.

Rest 5 elements can be written in $(5-1)! = 24$ ways.

$$\begin{aligned}\text{Total number of elements} &= {}^{10}C_2 \times 1 + {}^{8}C_3 \times 2 + 1 \times 24 \\ &= \frac{10\times 9}{2} + \frac{8\times 7\times 6}{3\times 2}\times 2 + 24 \\ &= 45 + 56 + 24 = 135 \text{ elements.}\end{aligned}$$

1.16 Let G be the set of all real 2×2 matrices $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$, where $xz \neq 0$. Show that G group under matrix multiplication. Let N denote the subset $\left\{\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \middle| a \in \mathbb{R}\right\}$. Is N a normal subgroup of G? Justify your answer.

(2014 : 10 Marks)

Solution:

Let $(G, *)$ be an algebraic structure. Where I∗I implies multiplication between its element in 'G' is the set of all 2×2 matrices of type $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$; $xz \neq 0$, $x, y, z \in R$

$$G = \left\{\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}; xz \neq 0; x, y, z \in R\right\}$$

(i) Closure Property:

$$\forall \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}, \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in G$$

$$x, y, z, a, b \in 1R \quad \text{...(i)}$$

$$xz \neq a, ac \neq 0 \quad \text{...(ii)}$$

$$\Rightarrow \quad \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} * \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} ax & xb + yc \\ 0 & zc \end{bmatrix} = A$$

$ax \in 1R$

$$(ax).(cz) = (ac)(xz) \neq 0 \text{ from (ii)}$$

$xb + yc \in 1R$

$$zc \in 1R \text{ from (i)}$$

$$\therefore \quad A \in G \Rightarrow (G, *) \text{ is closed}$$

(ii) Associative Property :

$$\forall \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}, \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, \begin{bmatrix} p & a \\ 0 & r \end{bmatrix} \in G$$

$$x, y, z, a, b, c, p, r \in 1R$$

$$xz \neq 0, ac \neq 0, Pr \neq 0$$

$$\Rightarrow \quad \left(\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} * \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) * \begin{bmatrix} p & a \\ 0 & r \end{bmatrix} = \begin{bmatrix} ax & bx + yc \\ 0 & zc \end{bmatrix} * \begin{bmatrix} p & a \\ 0 & r \end{bmatrix}$$

$$= \begin{bmatrix} apx & aqx + bxr + ycr \\ 0 & zcr \end{bmatrix} \quad \text{...(iii)}$$

$$\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} * \left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} * \begin{bmatrix} p & q \\ 0 & r \end{bmatrix}\right) = \begin{bmatrix} apx & aqx + bxr + ycr \\ 0 & zcr \end{bmatrix} \quad \text{...(iv)}$$

Here, (iii) = (iv)

i.e. $$\left(\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} * \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) * \begin{bmatrix} p & q \\ 0 & r \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} * \left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} * \begin{bmatrix} p & a \\ 0 & r \end{bmatrix}\right)$$

$\therefore$ (G, x) satisfy associative property.

(iii) Existence of Left Identity :

Let $$\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \in G, \text{ where } xz \neq 0, x, y, z \in 1R \quad \text{...(v)}$$

Let $$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in G, \text{ where } ac \neq 0, a, b, c \in 1R \quad \text{...(vi)}$$

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} * \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$$

$$\begin{bmatrix} ax & ay + bz \\ 0 & cz \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$$

equating its element, $cz = z \Rightarrow z(c-1) = 0$

From (vi), $ac \neq 0 \Rightarrow a \neq 0.\ c \neq 0$

From (v) $xz \neq 0 \Rightarrow x \neq 0, z \neq 0$

$\therefore$ $c - 1 = 0 \Rightarrow c = 1$

$ax = x \Rightarrow x(1 - a) = 0$

From (v) $x \neq 0$ $\quad 1 - a = 0 \qquad a = 1$

$$ay + bz = y \qquad y + bz = y$$

$\Rightarrow$ $\quad bz = 0$ $\qquad$ From (v) $\quad z \neq 0$

$\therefore$ $\quad b = 0$ $\qquad \therefore \quad \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow$ left identity.

(iv) Existence of left inverse:

Let $\quad \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}, \begin{bmatrix} p & q \\ 0 & r \end{bmatrix} \in G$

Where, $\quad xz \neq 0,\ pr \neq 0$

$$x, y, z, p, q, r \in 1R$$

$$\neq 0$$

$$\begin{bmatrix} p & q \\ 0 & r \end{bmatrix} * \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} px & py + qz \\ 0 & rz \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

equating the elements,

$$px = 1 \qquad p = \frac{1}{x}$$

$$rz = 1 \qquad r = \frac{1}{z}$$

$$py + az = 0 \qquad \frac{y}{x} + az = 0$$

$$q = -\left(\frac{y}{xZ}\right)$$

$\therefore \quad \begin{bmatrix} p & q \\ 0 & r \end{bmatrix} = \begin{bmatrix} \frac{1}{x} & \frac{-y}{xZ} \\ 0 & \frac{1}{2} \end{bmatrix}$ left inverse of $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$ ...(vii)

$\therefore \quad (G, *)$ is a group,

$$N = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}; a \in 1R \right\} \subset G$$

$$e = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \in N;\ N \neq \phi$$

Let, $\quad A = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \in N;\ x \in 1R$

$$A^{-1} = \begin{bmatrix} \frac{1}{1} & -\frac{x}{1} \\ 0 & \frac{1}{1} \end{bmatrix} = \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} \qquad \text{From (vii)}$$

(i) $A^{-1} \in N$

(ii) Let, $\quad B = \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} \in N;\ y \in 1R$

$$A * B = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & y + x \\ 0 & 1 \end{bmatrix} \in N$$

$\therefore \qquad N < G \quad$ ('N' is subgroup of G).

For 'N' to be a normal subgroup.

$$\forall X \in G \Rightarrow X * n * X^{-1} \in N$$
$$n \in N$$

Let,
$$x = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}, x^{-1} = \begin{bmatrix} \frac{1}{x} & \frac{-y}{xz} \\ 0 & \frac{1}{z} \end{bmatrix}$$

$$XnX^{-1} = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{x} & \frac{-y}{xz} \\ 0 & \frac{1}{z} \end{bmatrix}$$

$$= \begin{bmatrix} x & xx+y \\ 0 & z \end{bmatrix} * \begin{bmatrix} \frac{1}{x} & \frac{-y}{xz} \\ 0 & \frac{1}{z} \end{bmatrix} = \begin{bmatrix} 1 & \frac{-xy}{xz} + \frac{xx}{z} + \frac{y}{z} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{xx}{z} \\ 0 & 1 \end{bmatrix} \in N$$

$\therefore N \trianglelefteq G$. (Normal subgroup of G).

1.17 (i) How many generators are there of cyclic group G of order 8. Explain.

(2015 : 5 Marks)

(ii) Taking a group $\{e, a, b, c\}$ of order 4, where e is the identity, construct composition tables showing that one is cyclic while other is not.

(2015 : 5 Marks)

Solution:

(i) Let a be a generator of group G with order 8. $\therefore$ a^i is generator of G if i and 8 are co-prime numbers, co-prime to 8 are 1, 3, 5, 7.

$\therefore$ Number of generators of G are 4.

(ii) **Cyclic Group :** Let a be the generator, such that $a^1 = a$, $a^2 = b$, $a^3 = c$ and $a^4 = e$.

$\therefore$ Table is

x	e	a	b	c
e	e	a	b	c
a	a	a^2	$ab = a^3$	$ac = a^4 = e$
b	b	$ba = a^3$	$b^2 = e$	$bc = a^5 = a$
c	c	$ca = e$	$cb = a$	$c^2 = a^2$

So, table is

x	e	a	$b = a^2$	$c = a^2$
e	e	a	a^2	a^3
a	a	a^2	a^3	e
$b = a^2$	a^2	a^3	e	a
$c = a^3$	a^3	e	a	a^2

The above table is a group as e occurs in every row and column. Further, transpose of elements does not change the table.

Non-Cyclic : Let $a = \{e, a, b, c\}$ such that $a^2 = b^2 = c^2 = e$.

$\therefore$ Table is

x	e	a	b	c
e	e	a	b	c
a	a	e	ab	ac
b	b	ab	e	bc
c	c	ac	bc	e

Above table is a group as e is present in each row and column.

1.18 Let p be a prime number and z_p denote the additive group of integers modulo p. Show that every non-zero element of z_p generates z_p.

(2016 : 15 Marks)

Solution:

Let z_p the additive group, where p is a prime number.

$$\therefore \quad z_p = \{0, 1, 2, 3, \ldots.. p-1\}$$

Let $a \neq 0 \in z_p$ and a be the group formed by a.

$$a = \langle a \rangle = \{a, 2a, 3a, \ldots..\}$$

Now, a will form a subgroup of z_p.

By Lagrange's theorem

$$0(a) \mid 0(z_p)$$

Here, $$0(z_p) = p = \text{prime number}$$

$$\therefore \quad 0(a) = 1 \text{ or } 0(a) = p$$

as $a \neq 0$, $0(a) \neq 1$

$$\therefore \quad 0(a) = p$$

$$\Rightarrow \quad a = z_p$$

$\therefore$ a generates z_p.

So, every non-zero element of z_p generates z_p.

1.19 Let G be a group of order n. Show that G is isomorphic to a subgroup of the permutation group S_n.

(2017 : 10 Marks)

Solution:

S_n = Group of all permutations of set G. For any $a \in G$, define a mapping.

$f_a : G \to G$ s.t. $f_a(x) = ax$, then

f_a is well defined as $x = y \Rightarrow f_a(x) = f_a(y)$.

f_a is one-one as $f_a(x) = f_a(y) \Rightarrow ax = ay \Rightarrow x = y$

Also, for any $y \in G$, since $f_a(a^{-1}y) = a(a^{-1}y) = y$, we find $a^{-1}y$ is pre-image of $y \Rightarrow f_a$ is onto.

Hence, f_a is permutation on G. $\therefore$ $f_a \in S_n$.

Let K be the set of all such permutations.

To show, $K \subseteq S_n$: $\quad K \neq \phi$ as $f_e \in K$.

Let $f_a, f_b \in K$ be any members, then

$$f_b \circ f_{b^{-1}}(x) = f_b(f_{b^{-1}}(x)) = f_b(b^{-1}x) = b(b^{-1}x) = ex = f_e(x)$$

$$\therefore \quad f_{b^{-1}} = (f_b)^{-1} \qquad (f_e = I \text{ is identify of } S_n)$$

Also, $$(f_a \circ f_b)x = f_a(bx) = a(bx) = f_{ab}(x) \ \forall \ x$$